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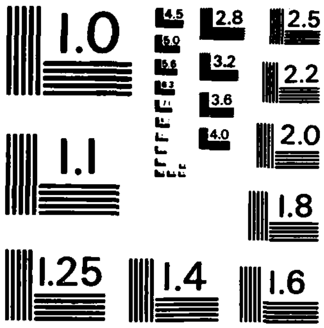

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ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

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ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

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ABSTRACT

For a positive constant λ we denote by $K(\lambda) = \{f(z)\}$ the class of function $f(z)$ which are regular and univalent in $|z| < \lambda$ and map this circle on a convex domain. In 1928 G. Szegö [3] proved the

<u>Theorem 1.</u> <u>If</u> $f(z) = \sum_0^\infty c_\nu z^\nu \in K(1)$, <u>then all its sections</u>

$$s_n(z) = \sum_0^n c_\nu z^\nu \in K\left(\frac{1}{4}\right) \quad \underline{\text{for}} \quad n = 1,2,\ldots .$$

An evident consequence is

<u>Corollary 1.</u> <u>Since the geometric series</u>

$$f_0(z) = \sum_0^\infty z^\nu \in K(1)$$

<u>we have</u>

$$\sum_0^n z^\nu \in K\left(\frac{1}{4}\right), \quad (n = 1,2,\ldots)$$

<u>and therefore, on replacing</u> z <u>by</u> $z/4$, <u>we have</u>

$$\sum_0^n \frac{1}{4^\nu} z^\nu \in K(1) \quad \underline{\text{for}} \quad n = 1,2,\ldots .$$

In the present paper we prove directly Corollary 1, and derive from it Szegö's Theorem 1. This is done by appealing to Theorem 2 which was conjectured by Polya and Schoenberg [1] in 1958, but only proved in 1973 by St. Ruscheweyh and T. Sheil-Small [2].

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SIGNIFICANCE AND EXPLANATION

There is a fine interplay between two fundamental notions of geometry: Convexity and Conformal Mapping. The subject belongs to Geometric Function Theory. In 1928 Gabor Szegö showed that if a power series converges in the unit circle $|z| < 1$ and maps it onto a convex domain, then all its finite sections map the circle $|z| < \frac{1}{4}$ onto convex domains. The present paper shows that Szegö's theorem reduces to a study of the finite sections of the geometric series

– 1/16 z squared 1/4 to the n power z to the n power

$$1 + \frac{1}{4} z + \frac{1}{4^2} z^2 + \ldots = \frac{1}{4^n} z^n + \ldots .$$

The main tool is a result conjectured in 1958 by Polya and Schoenberg, but only established in 1973 by St. Ruscheweyh and T. Sheil-Small.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

I. J. Schoenberg

1. INTRODUCTION. There is an interesting interplay between two fundamental notions of geometry: Convexity and Conformal Mapping. The subject belongs to Geometric Function Theory. We use the following notation: If $r > 0$, we denote $K(r) = \{f(x)\}$ the class of functions $f(z)$ which are univalent in the circle $|z| < r$, and map it onto a convex domain.

In 1958 Polya and Schoenberg conjectured that for $r = 1$ the class $K(1)$ form a semi-group with respect to Hadamard multiplication of power series. This was established in 1973 by St. Ruscheweyh and T. Sheil-Small in [2] by the following

Theorem 1. (Ruscheweyh and Sheil-Small). If

$$\sum_0^\infty a_\nu z^\nu \in K(1) \quad \text{and} \quad \sum_0^\infty b_\nu z^\nu \in K(1)$$

then

$$\sum_0^\infty a_\nu b_\nu z^\nu \in K(1) .$$

Before we pass to the work of Szegö, let us use Theorem 1 to establish a simple and well known

Proposition. If

$$f(z) = \sum_0^\infty a_\nu z^\nu \in K(1) \tag{1.1}$$

and $0 < \lambda < 1$, then

$$f(z) = \sum_0^\infty a_\nu z^\nu \in K(\lambda) . \tag{1.2}$$

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Proof: We start from the geometric series

$$f_0(z) = \sum_0^\infty z^\nu = \frac{1}{1 - z} \in K(1) \,, \tag{1.3}$$

because it maps $|z| < 1$ onto the half-plane $\operatorname{Re} z > \frac{1}{2}$. However, $w = f_0(z) = 1/(1 - z)$ clearly maps $|z| < \lambda$ onto a circle, and so

$$\sum_0^\infty z^\nu \in K(\lambda) \,.$$

Replacing z by λz, we obtain that

$$\sum_0^\infty \lambda^\nu z^\nu \in K(1) \,. \tag{1.4}$$

However, from (1.1) and (1.4), by Theorem 1 we obtain

$$\sum_0^\infty \lambda^\nu a_\nu z^\nu \in K(1) \,,$$

or, replacing λz by z, we have

$$\sum_0^\infty a_\nu z^\nu \in K(\lambda) \,,$$

which is the desired conclusion (1.2). ∎

Remark. Our derivation of the conclusion (1.2) from the special case of the geometric series (1.3) justifies Polya's statement that within the class $K(1)$ the geometric series (1.3) "sets the fashion" (in German: "tonangebend").

Szegö's question: If

$$f(z) = \sum_0^\infty c_\nu z^\nu \in K(1) \,, \tag{1.5}$$

what can we say about its sections

$$S_n(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \ ?$$

In 1928 G. Szegö [3, Satz II', page 204] established

Theorem 2 (Szegö). If (1.5) holds, then

(1.6) $$S_n(z) = \sum_0^n c_\nu z^\nu \in K(\tfrac{1}{4}) \quad \text{for} \quad n = 1,2,\ldots .$$

Using Theorem 1 we will show in §2 that we can reduce Theorem 2 to the question concerning the section of the geometric series

$$f_0(z) = 1 + \frac{1}{4} z + \ldots + \frac{1}{4^n} z^n + \ldots .$$

Theorem 3. Let λ be a constant satisfying

$$0 < \lambda < 1 .$$

If (1.5) holds, then

(1.7) $$S_n(z) = \sum_0^n c_\nu z^\nu \in K(\lambda) \quad \text{for all} \quad n \geqq m(\lambda) ,$$

where

(1.8) $m(\lambda)$ is the least integer such that (1.7) holds.

Evidently

(1.9) $$m(\lambda) = 1 \quad \text{if} \quad \lambda \leqq \frac{1}{4} ,$$

by Theorem 2. It is equally evident that

(1.10) $$\lambda_1 < \lambda_2 \quad \text{implies that} \quad m(\lambda_1) \leqq m(\lambda_2) ,$$

for if $S_n(z) \in K(\lambda_2)$, then also $S_n(z) \in K(\lambda_1)$.

The main difficulty in Theorem 3 is an explicit determination of $m(\lambda)$, if $\lambda > \frac{1}{4}$, and we will do that for the value

(1.11) $$\lambda = \frac{1}{3}$$

only, and find that

(1.12) $$m(\tfrac{1}{3}) = 4 .$$

We state this result as

Theorem 4. If

$$(1.13\qquad f(z) = \sum_0^{\infty} c_\nu z^\nu \in K(1)$$

then

$$(1.14)\qquad S_n(z) = \sum_0^{n} c_\nu z^\nu \in K\left(\tfrac{1}{3}\right) \quad \text{for} \quad n \geq 4 ,$$

but not necessarily for $n = 2$ or 3.

In establishing Theorems 3 and 4 I greatfully acknowledge the help of Fred W. Sauer, of the Computing Staff of the Mathematics Research Center.

2. THE NEW APPROACH TO SZEGÖ'S THEOREM 2.

Since evidently

$$(2.1)\qquad f_0(z) = \sum_0^{\infty} z^\nu \in K(1)$$

we conclude by Szegö's Theorem 2 that

$$(2.2)\qquad \sum_0^{n} z^\nu \in K\left(\tfrac{1}{4}\right) \quad \text{for all} \quad n = 1,2,\ldots .$$

Replacing z by $z/4$ we may restate (2.2) as

Corollary 1. We have

$$(2.3)\qquad s_n^{(0)}(z) = \sum_0^{n} \frac{1}{4^\nu} z^\nu \in K(1) \quad \underline{\text{for}} \quad n = 1,2,\ldots ,$$

The new approach to Theorem 1 is to establish Corollary 1 *directly*. If now

$$(2.4)\qquad f(z) = \sum_0^{\infty} c_\nu z^\nu \in K(1)$$

is an arbitrary element of $K(1)$ we argue as follows: Applying Theorem 1 to (2.3) and

(2.4) we conclude that

$$\sum_{\nu=0}^{n} \frac{1}{4^\nu} c_\nu z^\nu \in K(1)$$

and therefore

$$\sum_{\nu=0}^{n} c_\nu z^\nu \in K\left(\frac{1}{4}\right) \quad \text{for} \quad n = 1,2,\ldots , \tag{2.5}$$

which is the conclusion (1.6) of Theorem 2.

Szegö's theorem is therefore made to depend on the proof that all sections of the geometric series

$$f(z) = \sum_{0}^{\infty} \frac{1}{4^\nu} z^\nu , \quad (|z| < 4) \tag{2.6}$$

are in $K(1)$.

3. A DIRECT PROOF OF COROLLARY 1. We begin with

Lemma 1. All sections

$$f_n(z) = \sum_{0}^{n} \frac{1}{4^\nu} z^\nu \quad (n = 1,2,\ldots) \tag{3.1}$$

are univalent in the closed unit circle $\bar{U} = \{|z| \leq 1\}$

Proof: We are to show that

$$|z| \leq 1, \quad |z_2| \leq 1, \quad z_1 \neq z_2 \ \text{imply that} \ f_n(z_1 \neq f(z_2) . \tag{3.2}$$

Observe that

$$|f_n(z_1) - f_n(z_2)| = |\sum_{1}^{n} \frac{1}{4^\nu} (z_1^\nu - z_2^\nu)| \tag{3.3}$$

$$= \frac{|z_1 - z_2|}{4} \cdot \left| 1 + \frac{z_1 + z_2}{4} + \frac{z_1^2 + z_1 z_2 + z_2^2}{4^2} + \ldots + \frac{z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1}}{4^{n-1}} \right|$$

However, by (3.2),

$$\left|\frac{z_1 + z_2}{4} + \dots + \frac{z_1^{n-1} + \dots + z_2^{n-1}}{4^{n-1}}\right| \leq \frac{2}{4} + \frac{3}{4^2} + \dots + \frac{n}{4^{n-1}} = \frac{7}{9} - \frac{3n + 4}{9\cdot 4^{n-1}} ,$$

the last equality being easily obtained by summation, or by complete induction. The last member being < 1. We obtain from (3.3) that

$$|f_n(z_1) - f_n(z_2)| > \frac{|z_1 - z_2|}{4} \left|1 - \left(\frac{7}{9} - \frac{3n + 4}{9\cdot 4^{n-1}}\right)\right| = \frac{|z_1 - z_2|}{4} \left|\frac{2}{9} + \frac{3n + 4}{9\cdot 4^{n-1}}\right| > 0 ,$$

and (3.2) is established.

Lemma 1 shows that the polynomial (3.1) maps the circle $|z| = 1$ onto a closed Jordan curve C_n. Separating real and imaginary parts by

(3.4) $$f_n(e^{it}) = x_n(t) + iy_n(t)$$

we obtain for C_n the parametric representation

(3.5) $$C_n : x = x_n(t),\ y = y_n(t), \quad (0 \leq t \leq 2\pi) .$$

Wishing to study its curvature

(3.6) $$\frac{1}{R} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

we first establish

<u>Lemma 2</u>. <u>Defining</u>

(3.7) $$T_n(t) = x_n'(t)y_n''(t) - y_n'(t)x_n''(t) ,$$

<u>we have</u>

(3.8) $$T_n(t) \geq 0 \quad \underline{\text{for}} \quad -\pi \leq t \leq \pi, \quad \underline{\text{and}} \quad n = 1,2,\dots .$$

<u>Proof</u>: From (3.1) and (3.4) we have

(3.9) $$x_n(t) = 1 + \sum_{\alpha=1}^{n} \frac{\cos \alpha t}{4^\alpha}, \quad y_n(t) = \sum_{\alpha=1}^{n} \frac{\sin \alpha t}{4^\alpha}$$

and (3.7) becomes

$$T_n(t) = \begin{vmatrix} x'_n & x''_n \\ y'_n & x''_n \end{vmatrix} = \begin{vmatrix} -\sum_1^n 4^{-\alpha}\alpha \sin \alpha t & -\sum_1^n 4^{-\beta}\beta^2 \cos \beta t \\ \sum_1^n 4^{-\alpha}\alpha \cos \alpha t & -\sum_1^n 4^{-\beta}\beta^2 \sin \beta t \end{vmatrix} .$$

By splitting the columns we rewrite the determinant as a sum of n^2 determinants obtaining

$$T_n(t) = \sum_{\alpha=1}^n \sum_{\beta=1}^n 4^{-\alpha-\beta}\alpha\beta^2(\sin \alpha t \sin \beta t + \cos \alpha t \cos \beta t) ,$$

and finally

$$T_n(t) = \sum_{\alpha,\beta=1}^n \frac{\alpha\beta^2}{4^{\alpha+\beta}} \cos(\alpha - \beta)t , \tag{3.10}$$

which is a cosine polynomial of order $n - 1$.

Establishing the non-negativity of a cosine polynomial is in general a difficult problem. Fortunately, in our case it is easy due to the structure of the infinite matrix

$$\left\| \frac{\alpha\beta^2}{4^{\alpha+\beta}} \right\|_{\alpha,\beta = 1,2,\ldots} \tag{3.11}$$

of the coefficients of all (3.10). For instance, we obtain $T_3(t)$ as the sum of the elements of the 3×3 matrix (3.11) provided with appropriate cosine factors:

$$T_3(t) = \begin{cases} \frac{1.1^2}{4^2} & + \frac{1.2^2}{4^3} \cos t + \frac{1.3^2}{4^4} \cos 2t \\ + \frac{2.1^2}{4^3} \cos t & + \frac{2.2^2}{4^4} \qquad + \frac{2.3^2}{4^5} \cos t \\ + \frac{3.1^2}{4^4} \cos 2t & + \frac{3.2^2}{4^5} \cos t + \frac{3.3^2}{4^6} \end{cases} .$$

Since $T_n(t)$ is a cosine polynomial, we may restrict t to $0 \leq t \leq \pi$.

For $n = 2$ (3.8) presents no difficulty since

$$T_2(t) = \frac{3}{32}(1 + \cos t) , \tag{3.13}$$

which is non-negative. I owe to Fred Sauer the positivity of $T_3(t)$, $T_4(t)$ and $T_5(t)$ who provided the following (positive) minima

$$\min_t T_3(t) = .01309$$

$$\min_t T_4(t) = .01221$$

$$\min_t T_5(t) = .01600 . \tag{3.14}$$

I claim that <u>the last result</u> (3.14) <u>allows us to show that</u>

$$\min_t T_n(t) > 0 \quad \underline{\text{for all}} \quad n \geq 6 , \tag{3.15}$$

but this requires some elementary Algebraic Analysis.

The sum of all elements of the infinite matrix (3.11) is

$$\sum_{\alpha;\beta=1}^{\infty} \frac{\alpha\beta^2}{4^{\alpha+\beta}} = \left(\sum_{\alpha=1}^{\infty} \frac{\alpha}{4^\alpha}\right)\left(\sum_{\beta=1}^{\infty} \frac{\beta^2}{4^\beta}\right) . \tag{3.16}$$

From this sum we subtract the sum of the elements of the principal $n \times n$ minor of (3.11), and define the new sequence

$$N_n = \left(\sum_1^{\infty} \frac{\alpha}{4^\alpha}\right)\left(\sum_1^{\infty} \frac{\beta^2}{4^\beta}\right) - \left(\sum_1^{n} \frac{\alpha}{4^\alpha}\right)\left(\sum_1^{n} \frac{\beta^2}{4^\beta}\right) . \tag{3.17}$$

By iteration of $z(d/dz)$ applied to $\sum_1^n (z/4)^\nu$ we obtain the identities

$$\sum_1^n \frac{\nu}{4^\nu} = \frac{4.4^n - 3n - 4}{9.4^n}, \quad \sum_1^n \frac{\nu^2}{4^\nu} = \frac{20.4^n - 9n^2 - 24n - 20}{27.4^n} , \tag{3.18}$$

and letting $n \to \infty$ we obtain

$$(3.19)\qquad \sum_1^\infty \frac{\alpha}{4^\alpha} = \frac{4}{9}, \quad \sum_1^\infty \frac{\beta^2}{4^\beta} = \frac{20}{27} .$$

Now the sequence (3.17) may be explicitly written as

$$(3.20)\qquad N_n = \frac{4}{9} \cdot \frac{20}{27} - \frac{4.4^n - 3n - 4}{9.4^n} \cdot \frac{20.4^n - 9n^2 - 24n - 20}{27.4^n} ,$$

which yield the numerical values

$$N_4 = .0216009195$$

and

$$(3.21)\qquad N_5 = .0073673303 < .0074 .$$

I claim that this last inequality completes our proof of Lemma 2: Indeed, from (3.21) and the definition (3.17) of N_5 we conclude that all $T_n(t)$ are positive for all $n > 5$, for in view of the relations (3.14) and (3.21) w have for $n > 5$ and all real t

$$T_n(t) > T_5(t) - N_5 \geqq \min_t T_5(t) - N_5 > .0160 - .0074 = .0086 > 0 .$$

4. PROOF OF THEOREM 3. Our previous discussion makes it clear that it suffices to consider the geometric series

$$(4.1)\qquad f(z) = \sum_0^\infty \lambda^\nu z^\nu = \frac{1}{1 - \lambda z} \qquad (|z| < \lambda^{-1}) ,$$

and prove

Lemma 3. For its partial sums we have

$$(4.2)\qquad S_n(z) = \sum_0^n \lambda^\nu z^\nu \in K(1) \quad \text{for all} \quad n \geqq m(\lambda) ,$$

where

(4.3) $m(\lambda)$ *is the least integer such that (4.2) holds.*

<u>Proof of Lemma 3</u>. Notice that

$$(4.4) \qquad f(e^{it}) = \frac{1}{1 - \lambda e^{it}} = x(t) + iy(t)$$

traces out a circle C having the interval of reals $[1/(1 + \lambda), 1/(1 - \lambda)]$ as diameter and therefore the radius

$$(4.5) \qquad R = \frac{\lambda}{1 - \lambda^2} .$$

Setting

$$(4.6) \qquad s_n(e^{it}) = x_n(t) + iy_n(t) ,$$

and observing that $|z| = 1$ is well within the circle of convergence of (4.1), it should be clear that the real periodic functions

$$x_n(t),\ y_n(t),\ x(t),\ y(t)$$

are regular in a neighborhood of the real t-axis. Also that $y_n(t)$ and $y_n(t)$, as well as their derivatives, converge uniformly to the corresponding derivatives of $x(t)$ and $y(t)$, respectively. It follows that the closed curve $C_n = (x_n(t), y_n(t))$ converges to the circle C and that its curvature

$$\frac{x_n'(t)y_n''(t) - y_n'(t)x_n''(t)}{(x_n'(t)^2 + y_n'(t)^2)^{3/2}}$$

converges uniformly in t, as $n \to \infty$, to the curvature $1/R$ of C. This establishes Lamma 3.

The determination of $m(\lambda)$, satisfying (4.3), is difficult and will be solved for $\lambda = 1/3$ only.

5. <u>THE CASE</u> $\lambda = \frac{1}{3}$: <u>PROOF OF THEOREM 4</u>. As an analogue of Lemma 1 we should prove that the image of $|z| = 1$ by

$$(5.1) \qquad f_n(z) = \sum_0^n \frac{1}{3^\nu} z^\nu \qquad (n = 1,2,\ldots)$$

are all simple closed curves. However, our simple proof of Lemma 1 does not generalize. Rather we consider

$$w = f_n(z) - 1 = \sum_1^n \frac{1}{3^\nu} z^\nu = \frac{z}{3} \frac{1 - (z/3)^n}{1 - (z/3)} \tag{5.2}$$

and show that it maps $z = e^{it}$ into a curve which is star-shaped with respect to the origin. This requires two facts: 1°. That as t varies from $-\pi$ to $+\pi$, the argument of the function (5.2) increases by 2π. 2°. That we have

$$\operatorname{Im}\left(\frac{\partial w}{dt} / w\right) > 0 \quad \text{for all } t . \tag{5.3}$$

However, we omit the tedious calculations.

We rather pass to considering the curvature of the curve; here matters are very close to those of §3 and obtained from them by replacing $\frac{1}{4}$ by $\frac{1}{3}$. We shall also use the same notations.

Setting

$$f_n(e^{it}) = x_n(t) + iy_n(t) , \tag{5.4}$$

we have as an analogue of Lemma 2 the

<u>Lemma 3</u>. <u>Defining</u>

$$T_n(t) = x_n'(t)y_n''(t) - y_n'(t)x_n''(t) , \tag{5.5}$$

<u>we have</u>

$$T_n(t) > 0 \quad \underline{\text{for}} \quad -\pi \leqq t \leqq \pi \quad \underline{\text{and}} \quad n \geqq 4, \tag{5.6}$$

<u>but not for</u> $n = 2$ <u>and</u> $n = 3$.

<u>Proof:</u> For the analogues of (3.10) and (3.17) we find

$$T_n(t) = \sum_{\alpha,\beta=1}^{n} \frac{\alpha\beta^2}{3^{\alpha+\beta}} \cos(\alpha - \beta)t \tag{5.7}$$

and

$$N_n = \Big(\sum_1^\infty \frac{\alpha}{3^\alpha}\Big)\Big(\sum_1^\infty \frac{\beta^2}{3^\beta}\Big) - \Big(\sum_1^n \frac{\alpha}{3^\alpha}\Big)\Big(\sum_1^n \frac{\beta^2}{3^\beta}\Big) \tag{5.8}$$

and explicitly

$$N_n = \frac{9}{8} - \left(\frac{3^{n+1} - 2n - 3}{3^n \cdot 4}\right)\left(\frac{3^{n+1} - n^2 - 3n - 3}{3^n \cdot 2}\right) . \tag{5.9}$$

Rounding off Fred Sauer's values to six decimal places, we have

$$\min_t T_2(t) = -.012346$$

$$\min_t T_3(t) = -.002057$$

$$\min_t T_4(t) = .004268$$

$$\min_t T_5(t) = .013733$$

$$\min_t T_6(t) = .014480, \qquad N_6 = .036836$$

$$\min_t T_7(t) = .017663, \qquad N_7 = .015400 . \tag{5.10}$$

The first two minima being negative shows that the curves $C_2 = (x_2(t), y_2(t))$ and $C_3 = (x_3(t), y_3(t))$ are not convex. However, <u>from (5.10) we can conclude that (5.6) holds</u>: From the above data we see that $T_4(z)$, $T_5(t)$, $T_6(t)$, and $T_7(t)$ are everywhere positive. Now (5.10) show that if $n > 7$ then

$$T_n(t) \geqq T_7(t) - N_7 \geqq \min_t T_7(t) - N_7 > .0176 - .0155 = .0021 > 0 .$$

Thus $T_n(t) \geqq 0$ for all t and $n = 4,5,\ldots,$ proving Theorem 4.

REFERENCES

1. G. Polya and I. J. Schoenberg, Remarks on de la Vallée Poussin means and convex conformal maps of the circle, Pacific J. of Mathematics 8 (1958), 295-334.

2. St. Ruscheweyh and T. Sheil-Small, Hadamard products of schlicht functions and the Polya-Schoenberg conjecture, Comment. Math. Helvet. 48 (1973), 119-135.

3. Gabor Szegö, Zur Theorie der schlichten Abbildungen, Math. Annalen, 100 (1928), 188-211.

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Convexity, Geometric Function Theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For a positive constant λ we denote by $K(\lambda) = \{f(z)\}$ the class of function $f(z)$ which are regular and univalent in $|z| < \lambda$ and map this circle on a convex domain. In 1928 G. Szegö [3] proved the

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ABSTRACT (cont.)

Theorem 1. If $f(z) = \sum_0^\infty c_\nu z^\nu \in K(1)$, then all its sections

$$s_n(z) = \sum_0^n c_\nu z^\nu \in K\left(\tfrac{1}{4}\right) \quad \text{for} \quad n = 1,2,\ldots .$$

An evident consequence is

Corollary 1. Since the geometric series

$$f_0(z) = \sum_0^\infty z^\nu \in K(1)$$

we have

$$\sum_0^n z^\nu \in K\left(\tfrac{1}{4}\right), \quad (n = 1,2,\ldots)$$

and therefore, on replacing z by $z/4$, we have

$$\sum_0^n \frac{1}{4^\nu} z^\nu \in K(1) \quad \text{for} \quad n = 1,2,\ldots .$$

In the present paper we prove directly Corollary 1, and derive from it Szegö's Theorem 1. This is done by appealing to Theorem 2 which was conjectured by Polya and Schoenberg [1] in 1958, but only proved in 1973 by St. Ruscheweyh and T. Sheil-Small [2].

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